Pseudo-Distance-Regularised Graphs Are Distance-Regular or Distance-Biregular

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Abstract

The concept of pseudo-distance-regularity around a vertex of a graph is a natural generalization, for non-regular graphs, of the standard distance-regularity around a vertex. In this note, we prove that a pseudo-distance-regular graph around each of its vertices is either distance-regular or distance-biregular. By using a combinatorial approach, the same conclusion was reached by Godsil and Shawe-Taylor for a distance-regular graph around each of its vertices. Thus, our proof, which is of an algebraic nature, can also be seen as an alternative demonstration of Godsil and Shawe-Taylor's theorem.

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1 Introduction

Distance-regularity around a vertex of a (regular) graph is the local analogue of distance-regularity. More precisely, a graph Γ with vertex set V is distance-regular around of a vertex u if the distance partition of V induced by u is regular (see, for instance, Brouwer, Cohen and Neumaier [3]). In [14], Godsil and Shawe-Taylor defined a distance-regularised graph as that being distance-regular around each of its vertices. The interest of these graphs relies on the fact that they are a common generalization of distance-regular graphs and generalized polygons. The authors of [14] used a combinatorial approach to prove that distance-regularised graphs are either distance-regular or distance-biregular. For some properties of the latter, see Delorme [7]. More recently, Fiol, Garriga and Yebra

[11] introduced the concept of pseudo-distance-regularity around a vertex, as a natural generalization for not necessarily regular graphs, of distance regularity around a vertex. In this note, we prove that the same conclusion obtained in [14] can be reached when Γ is *pseudo-distance-regularized*; that is, pseudo-distance-regular graph around each of its vertices. In fact, this was already obtained in [11], but making strong use of the result in [14]. Here we provide an independent direct proof, which is simple and of algebraic nature. Thus, our contribution can be seen as an alternative demonstration of Godsil and Shawe-Taylor's theorem. Moreover, it turns out that the same conclusion of the theorem is obtained from the seemingly weaker condition of pseudo-distance-regularity.

2 Preliminaries

Let us first give some basic notation and results on which our proof is based. For more background on graph spectra, distance-regular and pseudo-distance-regular graphs see, for instance, [1, 2, 3, 4, 5, 6, 8, 9, 11].

Throughout this note, Γ is a connected graph with vertex set $V = V(G)$, $n = |V|$ vertices, adjacency matrix $\boldsymbol{A}$ and spectrum $\operatorname{sp}\Gamma = \operatorname{sp}\boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where the different eigenvalues of Γ are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$, $i = 0, 1, \ldots, d$. Then, as it is well known, λ_0, with multiplicity $m_0 = 1$, coincides with the spectral radius of $\boldsymbol{A}$, and has a positive (column) eigenvector (the *Perron vector*) $\boldsymbol{\alpha}$, which we normalize in such a way that $\|\boldsymbol{\alpha}\|^2 = n$. Let $\operatorname{dist}(u, v)$ represent the distance between the vertices $u, v \in V$. Then, $\Gamma_i(u) = \{v | \operatorname{dist}(u, v) = i\}$, the *eccentricity* of a vertex u is $\operatorname{ecc}(u) = \max\{\operatorname{dist}(u, v) | v \in V\}$, and the *diameter* of Γ is $D = \max\{\operatorname{ecc}(u) | u \in V\}$. For every $0 \le i \le D$, the *distance-i matrix* $\boldsymbol{A}_i$ has entries $(\boldsymbol{A}_i)_{uv} = 1$ if $\operatorname{dist}(u, v) = i$, and $(\boldsymbol{A}_i)_{uv} = 0$, otherwise. We also use the *weighted distance-i matrix*, defined as $\boldsymbol{A}_i^* = \boldsymbol{A}_i \circ \boldsymbol{J}^*$, where $\boldsymbol{J}^* = \boldsymbol{\alpha}\boldsymbol{\alpha}^\top$ and '$\circ$' stands for the Hadamard product; that is, $(\boldsymbol{A}_i^*)_{uv} = \alpha_u \alpha_v$ if $\operatorname{dist}(u, v) = i$, and $(\boldsymbol{A}_i^*)_{uv} = 0$, otherwise.

2.1 Local spectrum and predistance polynomials

Given a graph Γ with adjacency matrix $\boldsymbol{A}$ and spectrum as above, its idempotents $\boldsymbol{E}_i$ correspond to the orthogonal projections onto the eigenspaces $\ker(\boldsymbol{A} - \lambda_i \boldsymbol{I})$, $i = 0, 1, \ldots, d$. Their entries $m_{uv}(\lambda_i) = (\boldsymbol{E}_i)_{uv}$ are called the *(crossed) uv-local multiplicities* of λ_i. In particular, the diagonal elements $m_u(\lambda_i) = m_{uu}(\lambda_i)$ are the so-called *u-local multiplicities* of λ_i, because they satisfy properties similar to those of the (global) multiplicities $m(\lambda_i)$, but when Γ is "seen" from the base vertex u. Indeed,

$$a_{uu}^{(\ell)} = (\boldsymbol{A}^\ell)_{uu} = \textstyle\sum_{i=0}^{d} m_u(\lambda_i)\lambda_i^\ell, \quad \text{and} \quad \textstyle\sum_{u \in V} m_u(\lambda_i) = m(\lambda_i). \tag{1}$$

The *u-local spectrum* of Γ is constituted by the eigenvalues of $\boldsymbol{A}$, say $\mu_0, \mu_1, \ldots, \mu_{d_u}$, with nonzero u-local multiplicity. Then, it is known that the vector space spanned by the

vectors $(\boldsymbol{A}^\ell)_u$, $\ell \geq 0$, (that is, the u-th columns of the matrices $\boldsymbol{A}^\ell$, $\ell \geq 0$) has dimension d_u, and the eccentricity of u satisfies $\mathrm{ecc}(u) \leq d_u$. When $\mathrm{ecc}(u) = d_u$ we say that u is *extremal* (for more details, see [11]).

An orthogonal base for such a vector space is the following. The *u-local predistance polynomials* $p_0^u, p_1^u, \ldots, p_{d_u}^u$, $\deg p_i = i$, associated to a vertex u of Γ with nonzero local multiplicities $m_u(\mu_i)$, $i = 0, 1, \ldots, d_u$, are a sequence of orthogonal polynomials with respect to the scalar product

$$\langle f, g\rangle_u = (f(\boldsymbol{A})g(\boldsymbol{A}))_{uu} = \sum_{i=0}^{d_u} m_u(\mu_i) f(\mu_i) g(\mu_i) = \sum_{i=0}^{d} m_u(\lambda_i) f(\lambda_i) g(\lambda_i),$$

normalized in such a way that $\|p_i^u\|_u^2 = \alpha_u^2 p_i^u(\lambda_0)$. We notice that in [11] the normalization condition was $\|p_i^u\|_u^2 = p_i^u(\lambda_0)$ but, although the theory remains unchanged, it seems more convenient to use the above. Then, in particular, $p_0^u = \alpha_u^2$ and $p_1^u = \frac{\alpha_u^2 \lambda_0}{\delta_u} x$. Indeed, they are orthogonal since $\langle 1, x\rangle_u = \sum_{i=0}^{d_u} m_u(\lambda_i)\lambda_i = 0$, and the normalization condition is fulfilled:

- $\|\alpha_u^2\|_u^2 = \alpha_u^4 \sum_{i=0}^{d} m_u(\lambda_i) = \alpha_u^4 = \alpha_u^2 p_0^u(\lambda_0)$.
- $\|\frac{\alpha_u^2\lambda_0}{\delta_u} x\|_u^2 = \frac{\alpha_u^4\lambda_0^2}{\delta_u^2} \sum_{i=0}^{d} m_u(\lambda_i)\lambda_i^2 = \frac{\alpha_u^4\lambda_0^2}{\delta_u} = \alpha_u^2 p_1^u(\lambda_0)$.

If Γ is δ-regular, then $\alpha_u = 1$, $\lambda_0 = \delta$, and we have $p_0 = 1$ and $p_1 = x$, which are the first distance polynomials for every distance-regular graph. More generally, and as expected, if Γ is distance-regular, the u-local predistance polynomials are independent of u and become the *distance polynomials* p_i, $i = 0, 1, \ldots, D$, satisfying

$$p_i(\boldsymbol{A}) = \boldsymbol{A}_i, \qquad i = 0, 1, \ldots, D. \tag{2}$$

As every sequence of orthogonal polynomials, the u-local predistance polynomials satisfy a three-term recurrence of the form

$$x p_i^u = b_{i-1}^* p_{i-1}^u + a_i^* p_i^u + c_{i+1}^* p_{i+1}^u, \qquad i = 0, 1, \ldots, d_u, \tag{3}$$

where $b_{-1}^* = c_{d_u+1}^* = 0$, and the other numbers b_{i-1}^*, a_i^*, and c_{i+1}^* are the Fourier coefficients of $x p_i^u$ in terms of p_{i-1}^u, p_i^u, and p_{i+1}^u, respectively.

2.2 Pseudo-distance-regularity around a vertex

Given a graph Γ as above, consider the mapping $\boldsymbol{\rho} : V \longrightarrow \mathbb{R}^n$ defined by $\boldsymbol{\rho}(u) = \alpha_u \boldsymbol{e}_u$, where $\boldsymbol{e}_u$ is the coordinate vector. Note that, since $\|\boldsymbol{\rho}(u)\| = \alpha_u$, we can see $\boldsymbol{\rho}$ as a function which assigns weights to the vertices of Γ. In doing so we "regularize" the graph, in the sense that the *average weighted degree* of each vertex $u \in V$ becomes a constant:

$$\delta_u^* = \tfrac{1}{\alpha_u} \textstyle\sum_{v\in\Gamma(u)} \alpha_v = \lambda_0, \tag{4}$$

where $\Gamma(u)=\Gamma_1(u)$. Using these weights, we consider the following concept. A partition $\mathcal{P}$ of the vertex set $V=V_1\cup\cdots\cup V_m$ is called *pseudo-regular* (or *pseudo-equitable*) whenever the *pseudo-intersection numbers*

$$b^*_{ij}(u)=\frac{1}{\alpha_u}\sum_{v\in\Gamma(u)\cap V_j}\alpha_v, \qquad i,j=0,1,\ldots,m \tag{5}$$

do not depend on the chosen vertex $u\in V_i$, but only on the subsets V_i and V_j. In this case, such numbers are simply written as b^*_{ij}, and the $m\times m$ matrix $\boldsymbol{B}^*=(b^*_{ij})$ is referred to as the *pseudo-quotient matrix* of $\boldsymbol{A}$ with respect to the (pseudo-regular) partition $\mathcal{P}$. Pseudo-regular partitions were introduced by Fiol and Garriga [10], as a generalization of the so-called regular partitions, where the above numbers are defined by $b^*_{ij}(u)=|\Gamma(u)\cap V_j|$ for $u\in V_i$. A detailed study of regular partitions can be found in Godsil [12] and Godsil and McKay [13].

Let u be a vertex of Γ with eccentricity $\mathrm{ecc}(u)=\varepsilon_u$. Then Γ is said to be *pseudo-distance-regular around* u (or *u-local pseudo-distance-regular*) if the *distance-partition* around u, that is $V=C_0\cup C_1\cup\cdots\cup C_{\varepsilon_u}$ where $C_i=\Gamma_i(u)$ for $i=0,1,\ldots,\varepsilon_u$, is pseudo-regular. From the characteristics of the distance-partition, it is clear that its pseudo-quotient matrix is tridiagonal $\boldsymbol{B}^*=(b^*_{ij})$ with nonzero entries $c^*_i=b^*_{i-1,i}$, $a^*_i=b^*_{i,i}$ and $b^*_i=b^*_{i+1,i}$, $0\le i\le\varepsilon_u$, with the convention $c^*_0=b^*_{\varepsilon_u}=0$. By (4), notice that $a^*_i+b^*_i+c^*_i=\lambda_0$ for $i=0,1\ldots,\varepsilon_u$. These parameters are called the *u-local (pseudo-)intersection numbers.* In [11], it was shown that local pseudo-distance regularity is a generalization of distance-regularity around a vertex. Indeed, if Γ is distance-regular around $u\in V$, with intersection numbers a_i,b_i,c_i, then the entries of the Perron vector $\boldsymbol{\alpha}$ have a constant value, say α^i, on each of the sets $\Gamma_i(u)$, $i=0,1,\ldots,\varepsilon_u$, and Γ turns out to be pseudo-distance-regular around u, with u-local intersection numbers

$$a^*_i=a_i,\quad b^*_i=\frac{\alpha^{i+1}}{\alpha^i}b_i,\quad c^*_i=\frac{\alpha^{i-1}}{\alpha^i}c_i,\qquad i=0,1,\ldots,\varepsilon_u. \tag{6}$$

Conversely, when the eigenvector $\boldsymbol{\alpha}$ of a pseudo-distance-regular graph Γ exhibits such a regularity (which is the case for all u when Γ is regular or bipartite biregular), we have that Γ is also distance-regular around u with intersection parameters given again by (6).

As happens for distance-regular graphs, the existence of the so-called *u-local distance polynomials*, satisfying the "local version" of (2), guarantees that Γ is pseudo-distance-regular around u.

Theorem 2.1 ([11]). *Let Γ be a graph having a vertex u with eccentricity ε_u. Then, Γ is pseudo-distance-regular around u if and only if the u-local predistance polynomials satisfy*

$$(p^u_i(\boldsymbol{A}))_u=(\boldsymbol{A}^*_i)_u,\qquad i=0,1,\ldots,\varepsilon_u.$$

Moreover, if this is the case, u is extremal, $\varepsilon_u=d_u$, and the u-local intersection numbers a^*_i ,b^*_i and c^*_i coincide with the Fourier coefficients of the recurrence (3).

By this result, note that the u-local intersection numbers are univocally determined, through the u-local predistance polynomials, by the u-local spectrum.

3 The proof

Now we are ready to give the algebraic proof that a graph which is pseudo-distance-regular around each of its vertices is either distance-regular or distance-biregular.

Theorem 3.1. *Every pseudo-distance-regularized graph* Γ *is either distance-regular or distance-biregular.*

Proof. Let v, w be two vertices adjacent to a vertex u. Then, by Theorem 2.1,

$$\alpha_u\alpha_v = (p_1^u(\boldsymbol{A}))_{uv} = \frac{\alpha_u^2\lambda_0}{\delta_u}(\boldsymbol{A})_{uv} = \frac{\alpha_u^2\lambda_0}{\delta_u}(\boldsymbol{A})_{uw} = (p_1^u(\boldsymbol{A}))_{uw} = \alpha_u\alpha_w,$$

and we infer that $\alpha_v = \alpha_w$ since $\alpha_u > 0$. Hence, since Γ is connected, all vertices at even (respectively odd) distance from u have component α_u (respectively α_v), and Γ is either bipartite biregular if $\alpha_u \neq \alpha_v$, or regular otherwise.

Moreover, by using the orthogonal decomposition of x^ℓ in terms of the base $\{p_i^u\}_{0\le i\le d_u}$, the number of ℓ-walks between two adjacent vertices u, v is

$$(\boldsymbol{A}^\ell)_{uv} = \sum_{i=0}^{d_u}\frac{\langle x^\ell, p_i^u\rangle_u}{\|p_i^u\|_u^2}(p_i^u(\boldsymbol{A}))_{uv} = \frac{\langle x^\ell, p_1^u\rangle_u}{\alpha_u^2 p_1^u(\lambda_0)}\alpha_u\alpha_v = \frac{\alpha_v}{\alpha_u}\frac{1}{\lambda_0}\sum_{i=0}^{d} m_u(\lambda_i)\lambda_i^{\ell+1} \tag{7}$$

and, similarly,

$$(\boldsymbol{A}^\ell)_{uv} = \frac{\alpha_u}{\alpha_v}\frac{1}{\lambda_0}\sum_{i=0}^{d} m_v(\lambda_i)\lambda_i^{\ell+1} \tag{8}$$

Hence, from (1), (7), and (8) we have that, given the local multiplicities of u, those of v are uniquely determined from the system

$$\begin{array}{rcll} \sum_{i=0}^d m_v(\lambda_i) & = & 1 & \\ \sum_{i=0}^d m_v(\lambda_i)\lambda_i^r & = & \frac{\alpha_u^2}{\alpha_v^2}\sum_{i=0}^d m_u(\lambda_i)\lambda_i^r, & \qquad r = 1, 2, \ldots, d, \end{array}$$

with d equations, d unknowns $m_v(\lambda_i)$ $(i = 1, 2, \ldots, d)$, and Vandermonde determinant. Then, we have only two possible cases:

1. If Γ is regular, then $\alpha_u = \alpha_v = 1$ for every $u, v \in V$ and, hence, every vertex has the same local spectrum (this is known to be equivalent to say that Γ is *walk-regular* [13]), and the same u-local distance-polynomials. Thus, Γ is distance-regular.

2. If Γ is bipartite (δ_1, δ_2)-biregular, then $\alpha_u = \sqrt{(\delta_1 + \delta_2)/2\delta_2}$ for every $u \in V_1$ and $\alpha_v = \sqrt{(\delta_1 + \delta_2)/2\delta_1}$ for every $v \in V_2$. In this case, every vertex of V_1 has the same local spectrum and the same holds for every vertex in V_2 (in this case, we could say that Γ is *walk-biregular*). Thus, the sequence of local u-predistance polynomials only depend on the partite set where u belongs to and, consequently, Γ is distance-biregular.

This completes the proof. □

As mentioned above, notice that, since every regular or biregular graph which is pseudo-distance-regular around a vertex is also distance-regular around such a vertex, Theorem 3.1 implies the result of Godsil and Shawe-Taylor in [14].

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